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Fringe thickness and maximum path length of binary trees

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Abstract

We show how to compute the maximum path length of binary trees with a given size and a given fringe thickness (the difference in length between a longest and a shortest root-to-leaf path). We demonstrate that the key to finding the maximum path length binary trees with size N and fringe thickness Δ is the height $h_{\Delta,N} = \lceil \log_2((N+1)(2^{\Delta}-1)/\Delta) \rceil$. First we show that trees with height $h_{\Delta,N}$ exist. Then we show that the maximum path length trees have height $h_{\Delta,N}-1$, $h_{\Delta,N}$, or $h_{\Delta,N}+1$. © 1998 Elsevier Science B.V. All rights reserved.

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1. Introduction

One measure of the efficiency of searching, inserting, and deleting in a class of trees is the average number of comparisons performed during these operations. The average number of comparisons is the average length of a path traversed in the tree by such an operation. Assuming a uniform probability distribution on the items stored in the tree, the average path length in a tree is the path length (the sum of the lengths of the paths from the root to each node in the tree) divided by its size (the number of nodes in the tree). Thus, the path length is a good measure of the usefulness of a class of trees.

Knuth [7] shows that the minimum path length binary trees with a given size N have all external nodes on two consecutive levels or on one level and have path length

$$(N+1)(\log_2(N+1)+1+\Theta-2^{\Theta}),$$

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where $\Theta = \lceil \log_2(N+1) \rceil - \log_2(N+1) \in [0,1)$. The maximum path length binary trees with a given size N are shaped like chains of nodes and the trees have path length

$$N(N+3)/2.$$

What happens in the gap between the $N \log N$ lower bound and the N^2 upper bound on the path length? In particular, what are the bounds on the path length if we have more information about the tree other than its size? Nievergelt and Wong [8] compute an upper bound for the path length of a binary tree given the size and the maximum size balance of the tree's subtrees. De Santis and Persiano [5] compute a lower bound for path length of binary trees with a given size and a given fringe thickness (the difference in length between a longest and a shortest path in the tree) and demonstrate that their bound is attainable when the fringe thickness is less than half of the size. Cameron and Wood [3] and, independently, De Prisco et al. [4] characterize the minimum path length binary trees for all sizes and fringe thicknesses. Klein and Wood [6] derive an upper bound

$$(N+1)(\log_2(N+1) + \Delta - \log_2 \Delta - 0.6623)$$

for a binary tree of size N and fringe thickness Δ. De Santis and Persiano [5] derive a tighter upper bound. Neither Klein and Wood nor De Santis and Persiano describe trees that exactly achieve the bound for all N and Δ. Cameron and Wood [2] give a description of the binary trees with the maximum path length for a given size, height, and fringe thickness, leaving open the question of which are the heights of the trees that achieve the maximum path length for a given fringe thickness and size.

In this paper, we answer the question that was left open by Cameron and Wood [2]. We show that the binary trees that have the maximum path length among all binary trees with size N and fringe thickness Δ have heights $h_{\Delta,N} - 1$ or $h_{\Delta,N}$ or $h_{\Delta,N} + 1$, where

$$h_{\Delta,N} = \left\lceil \log_2 \frac{(N+1)(2^{\Delta} - 1)}{\Delta} \right\rceil.$$

In Section 2, we give some definitions and previous results. In Section 3, we show that there exist trees of height $h_{\Delta,N}$ for all fringe thicknesses $\Delta \geqslant 2$ and all sizes $N > 4$. In Section 4, we compute the possible heights of maximum path length binary trees with a given fringe thickness and size. In Section 5, we conclude with some open problems.

2. Definitions

We work with *extended binary trees*, which have external and internal nodes. An *external node* has no children and is represented as a square in diagrams. An *internal node* has exactly two children and is represented as a circle in diagrams; see Fig. 1. The *size* of a binary tree is the number of internal nodes. The size and the number of

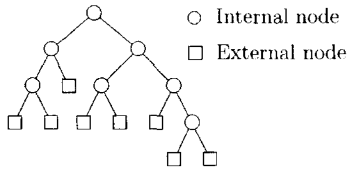

Fig. 1. An extended binary tree of size 7.

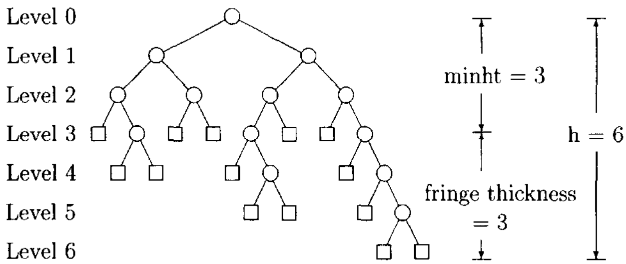

Fig. 2. An extended binary tree with height 6, minheight 3, and fringe thickness 3.

external nodes in a binary tree are related: a binary tree has size N if and only if it has $N+1$ external nodes.

Because we are interested in the path length of a binary tree, we need to know the length of (the number of edges on) the path from the root to each external node. The *level* of a node is the length of the path from the root of the tree to the node. Thus, the root is at Level 0, its children are at Level 1, and so on. The *height* of a binary tree T is denoted by h and defined to be the length of a longest path from the root to an external node. The *minheight* of T is denoted by $\mathrm{minht}(T)$ and defined to be the length of a shortest path from the root to an external node. The *fringe* of T is the set of external nodes of T. The *fringe thickness* of T is defined to be $h - \mathrm{minht}(T)$. These definitions are illustrated in Fig. 2.

The *(external) path length* of binary tree T is denoted by $\mathrm{EPL}(T)$ and is defined to be sum of the length of the paths from the root to each external node.

To calculate the external path length of tree T, the level of a node is important, but the connections between nodes are unimportant. For our purposes, the two trees pictured in Fig. 3 are equivalent since they have the same numbers of external nodes on each level, even though the trees are not isomorphic. Therefore, we will describe a binary tree T with fringe thickness Δ and size N by giving its height h and *fringe profile*. The fringe profile of a tree of fringe thickness Δ and height h is the sequence $\varepsilon_{h-\Delta}, \varepsilon_{h-\Delta+1}, \ldots, \varepsilon_{h-1}$, where ε_i is the number of external nodes on Level i. For example, the tree in Fig. 2 has fringe profile 5, 4, 3. In Section 4, when we compare two binary trees with a given fringe thickness Δ, a given size N, and different heights, we add the heights as a superscript to the fringe profiles to differentiate between them: the fringe profile of a tree with height h is Fringe Profile$(h) = \varepsilon^{(h)}_{h-\Delta}, \varepsilon^{(h)}_{h-\Delta+1}, \ldots, \varepsilon^{(h)}_{h-1}$.

Occasionally, we want a more detailed description of a tree that includes the numbers of internal nodes on each level of the tree. The *detailed profile* of binary tree T

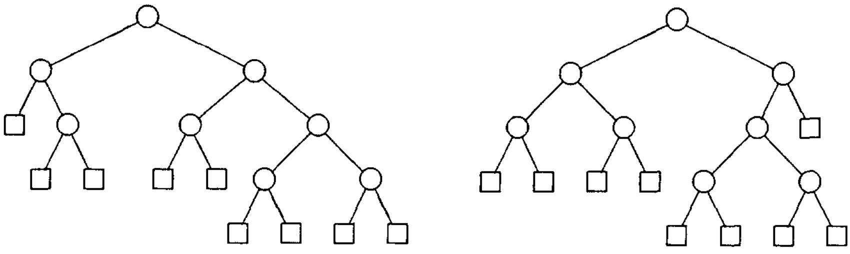

Fig. 3. Two non-isomorphic binary trees that, for our purposes, are equivalent.

is denoted by $\pi(T)$ and is defined to be $\pi(T) = \langle \iota_0, \varepsilon_0 \rangle, \ldots, \langle \iota_h, \varepsilon_h \rangle$, where ι_i is the number of internal nodes on Level i and ε_i is the number of external nodes on Level i. For example, the tree in Fig. 2 has detailed profile $\pi(T) = \langle 1,0 \rangle$, $\langle 2,0 \rangle$, $\langle 4,0 \rangle$, $\langle 3,5 \rangle$, $\langle 2,4 \rangle$, $\langle 1,3 \rangle$, $\langle 0,2 \rangle$. Clearly, given the detailed profile of a tree of size N and fringe thickness Δ, we can compute the height (one less than the number of pairs of integers in the detailed profile) and fringe profile (each of $\varepsilon_{h-\Delta}, \ldots, \varepsilon_{h-1}$ appears in the detailed profile). Given the height and fringe profile of a binary tree of fringe thickness Δ and size N, we can also compute the detailed profile:

- $\iota_0 = 1$, since the root is an internal node;
- $\varepsilon_i = 0$, for $0 \leqslant i < h - \Delta$, since $h - \Delta$ is the minheight of the tree;
- $\iota_h = 0$, since only external nodes appear on the last level;
- $\varepsilon_h = N + 1 - \sum_{i=h-\Delta}^{h-1} \varepsilon_i$, since there are $N + 1$ external nodes altogether and no external nodes on Levels $0, \ldots, h - \Delta - 1$; and
- since each of the nodes on Level $i+1$ must have an internal-node parent on Level i and an internal node on Level i must have two children on Level $i + 1$, we have $2\iota_i = \iota_{i+1} + \varepsilon_{i+1}$, for all $i \geqslant 0$. Using this equation, we can compute the following:
  (1) $\iota_i = 2^i$, for $0 < i < h - \Delta$;
  (2) $\iota_{h-\Delta} = 2^{h-\Delta} - \varepsilon_{h-\Delta}$; and
  (3) $\iota_{h-\Delta+1}, \ldots, \iota_{h-1}$ can be computed in that order from $\iota_{h-\Delta}$ and $\varepsilon_{h-\Delta+1}, \ldots, \varepsilon_{h-1}$ by $\iota_i = 2\iota_{i-1} - \varepsilon_i$.

In a *perfect binary tree* of height h (denoted by $\mathrm{Bin}(h)$), all external nodes are on Level h. The perfect binary trees are the only binary trees with fringe thickness 0.

A binary tree T has a *binary prefix* of height b (a $\mathrm{Bin}(b)$ prefix) if $\mathrm{minht}(T) = b$. If T has a binary prefix of height b, then the first b levels match those of a perfect binary tree of the same height and Level b contains at least one external node.

Part of the description of maximum path length binary trees of a given fringe thickness, height, and size involves the representation of an integer in the pseudo-binary number system. The *pseudo-binary number system* uses the digits 0, 1, and 2, and the ith digit of a pseudo-binary representation is the coefficient of $2^i - 1$. (The least significant digit corresponds to index 1, and we count up from there.) A pseudo-binary representation $a_k \cdots a_1$ is *canonical* when either none of its digits is two or exactly one digit a_j is 2 and all lower order digits a_i are 0, for $1 \leqslant i < j$. Cameron and Wood [1] show that every non-negative integer has exactly one canonical pseudo-binary representation and that this representation is the representation computed by the greedy algorithm.

Proposition 1 (Cameron and Wood [1]). *Let $a_m \cdots a_1$ be the canonical pseudo-binary representation of M. Then $M \leqslant 2(2^m - 1)$.*

Lemma 2. *Let $a_m \cdots a_1$ be the canonical pseudo-binary representation of M and let k satisfy $1 \leqslant k \leqslant m$. Then $a_i = 1$, for $k \leqslant i \leqslant m$, if and only if*

$$2^{m+1} - 2^k - m + k - 1 \leqslant M \leqslant 2^{m+1} - m + k - 3.$$

Proof. Suppose $a_i = 1$, for $k \leqslant i \leqslant m$. Then $M = \sum_{i=1}^{k-1} a_i(2^i - 1) + 2^{m+1} - 2^k - m + k - 1$. Since $a_{k-1} \cdots a_1$ is a canonical pseudo-binary number if $a_m \cdots a_1$ is, by Proposition 1, $0 \leqslant \sum_{i=1}^{k-1} a_i(2^i - 1) \leqslant 2(2^{k-1} - 1)$. Therefore, $2^{m+1} - 2^k - m + k - 1 \leqslant M \leqslant 2^{m+1} - m + k - 3$.

Let $b_m b_{m-1} \cdots b_1$ be the canonical pseudo-binary representation of Q. Suppose that at least one of $b_m, b_{m-1}, \ldots, b_k$ is not a 1. We show that Q does not fall within the range $[2^{m+1} - 2^k - m + k - 1, 2^{m+1} - m + k - 3]$; that is, we show that $Q < 2^{m+1} - 2^k - m + k - 1$ or that $2^{m+1} - m + k - 3 < Q$.

Let p be the largest index such that $k \leqslant p \leqslant m$ and that $b_p \neq 1$.

If $b_p = 0$, we show that $Q < 2^{m+1} - 2^k - m + k - 1$. We consider $p = m$ and $k \leqslant p < m$ separately.

If $p = m$, then $b_m b_{m-1} \cdots b_1 = 0 b_{m-1} \cdots b_1$. Since $b_{m-1} \cdots b_1$ is a canonical pseudo-binary representation if $b_m b_{m-1} \cdots b_1$ is, by Proposition 1, Q is maximized if $b_{m-1} \cdots b_1$ is $20 \cdots 0$; that is, $Q \leqslant 2(2^{m-1} - 1) = 2^m - 2$. Since $k \leqslant m$, we have $2^m - 1 \leqslant 2^{m+1} - 2^k - m + k - 1$. Therefore, $Q \leqslant 2^m - 2 < 2^m - 1 \leqslant 2^{m+1} - 2^k - m + k - 1$.

If $k \leqslant p < m$, then $b_m \cdots b_{p+1} b_p b_{p-1} \cdots b_1 = 1 \cdots 10 b_{p-1} \cdots b_1$. Since $b_{p-1} \cdots b_1$ is a canonical pseudo-binary representation if $b_m b_{m-1} \cdots b_1$ is, by Proposition 1, Q is maximized if $b_{p-1} \cdots b_1 = 20 \cdots 0$. Therefore, $Q \leqslant \sum_{i=p+1}^{m} (2^i - 1) + 2(2^{p-1} - 1) = 2^{m+1} - 2^p - m + p - 2$. Since $k \leqslant p$, we have $2^k - k \leqslant 2^p - p$. Therefore, $Q \leqslant 2^{m+1} - m - (2^p - p) - 2 < 2^{m+1} - m - (2^k - k) - 1$.

Therefore, if $b_p = 0$, then $Q < 2^{m+1} - 2^k - m + k - 1$.

If $b_p = 2$, then we show that $2^{m+1} - m + k - 3 < Q$. Since $b_p = 2$ and $b_m \cdots b_1$ is a canonical pseudo-binary representation, we have $b_i = 0$, for $1 \leqslant i < p$. We consider $p = m$ and $k \leqslant p < m$ separately.

If $p = m$, then $b_m \cdots b_1 = 20 \cdots 0$. Therefore, $Q = 2(2^m - 1) = 2^{m+1} - 2$. Since $1 \leqslant k \leqslant m$, we have $2^{m+1} - m + k - 3 < 2^{m+1} - 2 = Q$.

If $k \leqslant p < m$, then $b_m \cdots b_1 = 1 \cdots 120 \cdots 0$, with the 2 at index p. Therefore, $Q = \sum_{i=p+1}^{m} (2^i - 1) + 2(2^p - 1) = 2^{m+1} - m + p - 2$. Since $k \leqslant p$, $2^{m+1} - m + k - 3 < 2^{m+1} - m + p - 2 = Q$.

Therefore, if $b_p = 2$, then $2^{m+1} - m + k - 3 < Q$.

Thus, if at least one of the high-order $m - k + 1$ digits of $b_m \cdots b_1$ is not 1, then Q does not fall within the range $[2^{m+1} - 2^k - m + k - 1, 2^{m+1} - m + k - 3]$. $\square$

The set of binary trees with the maximum path length among all binary trees with fringe thickness Δ, height h, and size N is denoted by MaxPL(Δ, h, N). The set of binary trees with the maximum path length among all binary trees with fringe thickness Δ and size N is denoted by MaxPL(Δ, N). For brevity, where the context is clear, we also use MaxPL(Δ, h, N) and MaxPL(Δ, N) to represent the path length of the trees in the set.

The following proposition from Cameron and Wood [2] describes the detailed profile of the binary trees in MaxPL(Δ, h, N):

Proposition 3 (Cameron and Wood [2]). *Let* $\pi(T) = \langle \iota_0, \varepsilon_0 \rangle, \ldots, \langle \iota_h, \varepsilon_h \rangle$ *be the detailed profile of a binary tree* T *in* MaxPL(Δ, h, N). *Then*

- *For all* i, $0 \leqslant i < h - \Delta$, $\iota_i = 2^i$ *and* $\varepsilon_i = 0$;
- $\iota_{h-\Delta} = r_h$ *and* $\varepsilon_{h-\Delta} = 2^{h-\Delta} - r_h$, *where* $r_h = \lfloor (N - 2^{h-\Delta})/(2^\Delta - 1) \rfloor + 1$;
- $\varepsilon_{h-\Delta+1}\varepsilon_{h-\Delta+2}\cdots\varepsilon_{h-1}$ *(ignoring leading zeros) is the canonical pseudo-binary representation of* $num_h = 2^{h-\Delta} + r_h(2^\Delta - 1) - N - 1$;
- *For all* $h - \Delta < i < h$, ι_i *can be found by using* $2\iota_{i-1} = \iota_i + \varepsilon_i$, *once* $\iota_{h-\Delta}$ *and* ε_i *are known*; *and*
- $\iota_h = 0$ *and* $\varepsilon_h = N + 1 - (2^{h-\Delta} - r_h) - \sum_{i=1}^{\Delta-1} \varepsilon_{h-\Delta+i}$.

The (external) path length of a tree in MaxPL(Δ, h, N) *is*

$$(N+1)h - \Delta(2^{h-\Delta} - r_h) - \sum_{i=1}^{\Delta-1} (\Delta - i)\varepsilon_{h-\Delta+i}.$$

We examine the *path length difference*, the difference between the path lengths of MaxPL$(\Delta, h+1, N)$ and MaxPL(Δ, h, N), which we denote by PL Diff(h). Then

$$\begin{aligned}
\text{PL Diff}(h) &= \text{MaxPL}(\Delta, h+1, N) - \text{MaxPL}(\Delta, h, N) \\
&= N + 1 - \Delta(2^{h-\Delta} - (r_{h+1} - r_h)) \\
&\quad - \sum_{i=1}^{\Delta-1} (\Delta - i)(\varepsilon^{(h+1)}_{h+1-\Delta+i} - \varepsilon^{(h)}_{h-\Delta+i}),
\end{aligned}$$

where $r_h = \lfloor (N - 2^{h-\Delta})/(2^\Delta - 1) \rfloor + 1$ is the number of internal nodes on Level $h - \Delta$ of a tree in MaxPL(Δ, h, N) and $\varepsilon_i^{(h)}$ is the number of external nodes on Level i of a tree in MaxPL(Δ, h, N).

We will need to examine more closely the contribution of the summation to PL Diff(h), so we give the summation a name, the *fringe difference*, and denote it by Fringe Diff(h):

$$\text{Fringe Diff}(h) = \sum_{i=1}^{\Delta-1} (\Delta - i)(\varepsilon^{(h+1)}_{h+1-\Delta+i} - \varepsilon^{(h)}_{h-\Delta+i}).$$

Cameron and Wood [2] give necessary and sufficient conditions for the existence of a binary tree with a given size, height, and fringe thickness:

Proposition 4 (Cameron and Wood [2]). *Let $N > 4$ and $2 \leqslant \Delta < h \leqslant N$. Then there exists a binary tree with fringe thickness Δ, height h, and size N if and only if*

$$(2^{h-\Delta} - 1) + \Delta \leqslant N \leqslant (2^{h-\Delta} - 1)2^{\Delta}.$$

$(2^{h-\Delta} - 1) + \Delta$ is the size of the binary trees with height h and fringe thickness Δ that have the minimum size. These trees consist of a binary prefix of height $h - \Delta$ and one chain of length Δ rooted on level $h - \Delta$. Similarly, $(2^{h-\Delta} - 1)2^{\Delta}$ is the size of the binary trees with height h and fringe thickness Δ that have the maximum size. These trees consist of a binary prefix of height $h - \Delta$ and $2^{h-\Delta} - 1$ Bin(Δ) subtrees rooted on level $h - \Delta$.

If we solve for h in the inequalities of the proposition, we obtain the following necessary and sufficient condition for the existence of a binary tree with height h among all the binary trees with a given fringe thickness Δ and a given size N:

Corollary 5. *Let $N > 4$ and $2 \leqslant \Delta < h \leqslant N$. Then there exists a binary tree with height h among the binary trees of size N and fringe thickness Δ if and only if*

$$\log_2(N + 2^{\Delta}) \leqslant h \leqslant \Delta + \log_2(N - \Delta + 1).$$

Let $h_{\min}$ be the lower bound and $h_{\max}$ the upper bound on the valid heights of a binary tree of size N and fringe thickness Δ; that is, let $h_{\min} = \log_2(N + 2^{\Delta})$ and $h_{\max} = \Delta + \log_2(N - \Delta + 1)$.

In the following sections, we will show that the maximum path length among all trees of size N and fringe thickness $\Delta > 0$ is achieved by the trees in MaxPL$(\Delta, h_{\Delta,N} - 1, N)$ or MaxPL$(\Delta, h_{\Delta,N}, N)$ or MaxPL$(\Delta, h_{\Delta,N} + 1, N)$, where

$$h_{\Delta,N} = \left\lceil \log_2 \frac{(N+1)(2^{\Delta} - 1)}{\Delta} \right\rceil.$$

The cases when $\Delta = 0$ and $\Delta = 1$ are trivial. When $\Delta = 0$, $h_{\Delta,N}$ is undefined and a binary tree with $\Delta = 0$ exists only if the size is a power of two, in which case the tree is a perfect binary tree. A tree with $\Delta = 1$ exists only if the size is not a power of two, in which case it is the well-studied complete binary tree with height $h_{\Delta,N}$; see Knuth [7]. In what follows, we assume that $\Delta > 1$.

3. Trees of height $h_{\Delta,N}$ exist

On our way to proving that all trees in MaxPL(Δ, N) have heights in $\{h_{\Delta,N} - 1, h_{\Delta,N}, h_{\Delta,N} + 1\}$, we need to show that $h_{\Delta,N}$ is a valid height for binary trees with size N and fringe thickness Δ. Therefore, we must prove that $h_{\min} \leqslant h_{\Delta,N} \leqslant h_{\max}$, that is,

$$\log_2(N + 2^{\Delta}) \leqslant h_{\Delta,N} \leqslant \Delta + \log_2(N - \Delta + 1).$$

Lemma 6. *Let $N>4$ and $2\leqslant \Delta < N$. Then* $h_{\min} = \log_2(N+2^{\Delta}) \leqslant h_{\Delta,N}$.

Proof. We prove that $N+2^{\Delta} \leqslant (N+1)(2^{\Delta}-1)/\Delta$. By rearranging and multiplying by Δ, we see that we must prove that

$$f(\Delta,N)=(N-\Delta+1)2^{\Delta}-N-1-N\Delta \geqslant 0.$$

For any fixed Δ, we show that $f(\Delta,N)$ is non-negative for the smallest value of N and that $f(\Delta,N)$ is an increasing function of N.

Since $N>4$, for $\Delta=2$ or $\Delta=3$, the smallest value of N is $N=5$. For $\Delta=2$, $f(\Delta,5)=0$. For $\Delta=3$, $f(\Delta,5)=3$. For any fixed $\Delta>3$, the smallest possible value of N is $\Delta+1$. For $\Delta>3$, $f(\Delta,\Delta+1)=2^{\Delta+1}-\Delta^2-2\Delta-2$ is positive, since $f(\Delta,\Delta+1)=6$ when $\Delta=4$ and the derivative is positive for $\Delta>3$. Therefore, for any fixed Δ, $f(\Delta,N)$ is non-negative for the smallest value N.

The derivative of $f(\Delta,N)$ with respect to N is $2^{\Delta}-1-\Delta$ and is positive for all $\Delta \geqslant 2$. □

For the second half of the proof that there exists a binary tree with size N, fringe thickness Δ, and height $h_{\Delta,N}$, we prove that

$$\left\lceil \log_2 \frac{(N+1)(2^{\Delta}-1)}{\Delta} \right\rceil \leqslant \Delta + \log_2(N-\Delta+1) = h_{\max}$$

or that the range

$$\left[\frac{(N+1)(2^{\Delta}-1)}{\Delta}, 2^{\Delta}(N+1-\Delta)\right]$$

contains a power of two.

First we show that the range is valid.

Lemma 7. *Let N and Δ be integers satisfying $N>4$ and $2\leqslant\Delta<N$. Then*

$$\frac{(N+1)(2^{\Delta}-1)}{\Delta} < 2^{\Delta}(N+1-\Delta).$$

Proof. Rearranging and multiplying by Δ, we see that we must show that

$$g(\Delta,N)=2^{\Delta}\Delta(N+1-\Delta)-(N+1)(2^{\Delta}-1)>0.$$

We show that, for a fixed Δ, $g(\Delta,N)>0$ for the smallest value of N and that the derivative with respect to N is positive.

Since $N>4$, the smallest value of N is $N=5$ for $\Delta=2$. For $\Delta=2$ and $N=5$, $g(\Delta,N)=14$. For all $\Delta\geqslant 3$, the smallest value of N is at least $\Delta+1$ (the smallest value is $N=\Delta+2$ for $\Delta=3$). At $N=\Delta+1$, $g(\Delta,\Delta+1)=2^{\Delta}(\Delta-2)+\Delta+2$ which is positive for all $\Delta\geqslant 3$.

The derivative of $g(\Delta,N)$ with respect to N is $2^{\Delta}(\Delta-1)+1$ which is positive for all $\Delta\geqslant 2$. □

Table 1
For small N and Δ, $[(N+1)(2^{\Delta}-1)/\Delta, 2^{\Delta}(N+1-\Delta)]$ contains a power of two

N	Δ	$(N+1)(2^{\Delta}-1)/\Delta$	Power of two contained	$2^{\Delta}(N+1-\Delta)$
5	2	9	16	16
	3	14	16	24
	4	22.5	32	32
6	2	10.5	16	20
	3	$49/3 \approx 16.33$	32	32
	4	26.25	32	48
	5	43.4	64	64
7	2	12	16	24
	3	$56/3 \approx 18.66$	32	40
	4	30	32, 64	64
	5	49.6	64	96
	6	84	128	128

Lemma 8. *The range* $[(N+1)(2^{\Delta}-1)/\Delta, 2^{\Delta}(N+1-\Delta)]$ *contains a power of two.*

Proof. Table 1, which shows the powers of two contained for some small values of N and Δ, is used as the basis of induction proofs. In each case, $[(N+1)(2^{\Delta}-1)/\Delta, 2^{\Delta}(N+1-\Delta)]$ contains a power of two.

Next consider small values of Δ and all values of N greater than $\Delta+1$. If $\Delta=2$, then

$$\frac{(N+1)(2^{\Delta}-1)}{\Delta}=\frac{3}{2}(N+1)$$

and

$$2^{\Delta}(N+1-\Delta)=4(N-1).$$

We prove by induction on N that the range $R_2(N)=[(N+1)3/2, 4(N-1)]$ contains a power of two. The basis for the induction is displayed in Table 1. If $R_2(N)$ contains the same power of two as $R_2(N-1)=[N3/2, 4(N-2)]$, then we are done. Otherwise, consider $R_2(N)=[(N+1)3/2, 4(N-1)]=[N3/2+3/2, 4(N-2)+4]$. Since $R_2(N)$ does not contain the same power of two as $R_2(N-1)$, $[N3/2, N3/2+3/2)$ must contain a power of two. Also, since $N>4$, range $[N3/2, N3/2+3/2)$ does not contain the next larger power of two. Therefore, $2(N3/2+3/2)=3(N+1)$ is larger than the next larger power of two and $[N3/2+3/2, 3(N+1))$ contains the next larger power of two. If $3(N+1)\leqslant 4(N-1)$, then $R_2(N)$ contains the next larger power of two. We have $3(N+1)\leqslant 4(N-1)$ when $7\leqslant N$. The cases when $\Delta=3$, $\Delta=4$ and $\Delta=5$ are similar, so we omit them.

Now consider the case when $\Delta=N-1$. In this case,

$$\frac{(N+1)(2^{\Delta}-1)}{\Delta}=\frac{(\Delta+2)}{\Delta}(2^{\Delta}-1)$$

and

$$2^{\Delta}(N+1-\Delta)=2^{\Delta+1}.$$

Clearly, $[(2^{\Delta}-1)(\Delta+2)/\Delta, 2^{\Delta+1}]$ contains a power of two.

What if $5<\Delta<N-1$? Let

$$2^b \leqslant \frac{(N+1)(2^{\Delta}-1)}{\Delta} < 2^{b+1}.$$

We show that $2^{b+1} \leqslant 2^{\Delta}(N+1-\Delta)$. Since $2^b \leqslant (N+1)(2^{\Delta}-1)/\Delta$, we have

$$2^b \frac{\Delta}{N+1} + 1 \leqslant 2^{\Delta}.$$

Multiplying both sides by $(N+1-\Delta)$, we have

$$2^b \frac{\Delta(N+1-\Delta)}{N+1} + (N+1-\Delta) \leqslant 2^{\Delta}(N+1-\Delta).$$

We show that $2 \leqslant \Delta(N+1-\Delta)/(N+1)$, which gives us

$$2^{b+1} \leqslant 2^b \frac{\Delta(N+1-\Delta)}{N+1} + (N+1-\Delta) \leqslant 2^{\Delta}(N+1-\Delta).$$

Since $6 \leqslant \Delta$, we have $0 \leqslant \Delta - 6$. Adding Δ^2 to both sides, we get $\Delta^2 \leqslant (\Delta+3)(\Delta-2)$. Since $\Delta+1<N$, we have $\Delta+3 \leqslant N+1$ and $\Delta^2 \leqslant (N+1)(\Delta-2)$. Rearranging terms, we get

$$2 \leqslant \frac{\Delta(N+1-\Delta)}{(N+1)}.$$

Since

$$\frac{(N+1)(2^{\Delta}-1)}{\Delta} < 2^{b+1} \leqslant 2^{\Delta}(N+1-\Delta),$$

the range

$$\left[\frac{(N+1)(2^{\Delta}-1)}{\Delta}, 2^{\Delta}(N+1-\Delta)\right]$$

contains a power of two when $5<\Delta<N-1$. □

Therefore, $h_{\Delta,N}$ is always a valid height for a binary tree of size N and fringe thickness Δ.

4. The heights of MaxPL(Δ, N)

In this section, we show that the trees in MaxPL(Δ,N) have heights $h_{\Delta,N}-1$, $h_{\Delta,N}$, or $h_{\Delta,N}+1$ by showing that PL Diff(h) is a positive function for heights $h_{\min} \leqslant h < h_{\Delta,N}-1$ and that PL Diff(h) is a negative function for heights $h_{\Delta,N} < h < h_{\max}$. In Section 4.1, we derive some preliminary results to be used in the proof. In Section 4.2, we give the main result.

4.1. Preliminary results

First we derive some results which apply to both halves of the result (heights $h_{\min} \leqslant h < h_{\Delta,N} - 1$ and heights $h_{\Delta,N} < h < h_{\max}$). We take a closer look at the possible values of $r_{h+1} - r_h$ and of Fringe Diff(h). Then we derive a more careful description of MaxPL(Δ, h, N) when $\Delta > \log_2(N+1)$, which we use to narrow the range of possible values for Fringe Diff(h) when $\Delta > \log_2(N+1)$. Finally, we examine the heights of trees in MaxPL(Δ, N) when $N \leqslant 43$.

Consider $r_{h+1} - r_h$, where r_h is the number of internal nodes on Level $h - \Delta$ of a tree in MaxPL(Δ, h, N).

Lemma 9. *Let h and $h+1$ be the heights of binary trees with size N and fringe thickness Δ and maximum path length. Then*

$$r_{h+1} - r_h = -\left\lfloor \frac{2^{h-\Delta}}{2^\Delta - 1} \right\rfloor$$

or

$$r_{h+1} - r_h = -\left\lceil \frac{2^{h-\Delta}}{2^\Delta - 1} \right\rceil .$$

Proof. Since

$$r_{h+1} = \left\lfloor \frac{N - 2^{h+1-\Delta}}{2^\Delta - 1} \right\rfloor + 1,$$

$$\frac{N - 2^{h+1-\Delta}}{2^\Delta - 1} < r_{h+1} \leqslant \frac{N - 2^{h+1-\Delta}}{2^\Delta - 1} + 1.$$

Similarly,

$$\frac{N - 2^{h-\Delta}}{2^\Delta - 1} < r_h \leqslant \frac{N - 2^{h-\Delta}}{2^\Delta - 1} + 1.$$

Therefore,

$$\frac{N - 2^{h+1-\Delta}}{2^\Delta - 1} - \left(\frac{N - 2^{h-\Delta}}{2^\Delta - 1} + 1 \right) < r_{h+1} - r_h < \frac{N - 2^{h+1-\Delta}}{2^\Delta - 1} + 1 - \frac{N - 2^{h-\Delta}}{2^\Delta - 1}$$

or

$$\frac{-2^{h-\Delta}}{2^\Delta - 1} - 1 < r_{h+1} - r_h < \frac{-2^{h-\Delta}}{2^\Delta - 1} + 1.$$

Since $2 \leqslant \Delta < h$, $2^\Delta - 1$ (an odd number > 1) does not divide $2^{h-\Delta}$ (an even number > 0), and there are two integers between $-2^{h-\Delta}/(2^\Delta - 1) - 1$ and $-2^{h-\Delta}/(2^\Delta - 1) + 1$. The integers are

$$-\left\lfloor \frac{2^{h-\Delta}}{2^\Delta - 1} \right\rfloor$$

and

$$-\left\lceil \frac{2^{h-\Delta}}{2^{\Delta}-1} \right\rceil.$$

Since r_{h+1} and r_h are integers, $r_{h+1} - r_h$ must also be an integer. Therefore,

$$r_{h+1} - r_h = -\left\lfloor \frac{2^{h-\Delta}}{2^{\Delta}-1} \right\rfloor$$

or

$$r_{h+1} - r_h = -\left\lceil \frac{2^{h-\Delta}}{2^{\Delta}-1} \right\rceil. \qquad \square$$

Next let us examine the possible range of values for Fringe Diff(h) by considering what the fringe profiles of MaxPL(Δ, h, N) and MaxPL($\Delta, h+1, N$) could be.

Lemma 10. *Let $\Delta \geqslant 2$. Then*

$$-\frac{\Delta^2 - \Delta + 2}{2} \leqslant \text{Fringe Diff}(h) \leqslant \frac{\Delta^2 - \Delta + 2}{2}.$$

Proof. Since $\varepsilon^{(h)}_{h-\Delta+1}\varepsilon^{(h)}_{h-\Delta+2}\cdots\varepsilon^{(h)}_{h-1}$ is the canonical pseudo-binary representation of some positive integer, $\varepsilon^{(h)}_{h-\Delta+i}$ is either 0, 1, or 2. Furthermore, at most one $\varepsilon^{(h)}_{h-\Delta+k} = 2$, for some $0 < k < \Delta$, in which case $\varepsilon^{(h)}_{h-\Delta+i} = 0$, for all $k < i < \Delta$. Similarly, for $\varepsilon^{(h+1)}_{h-\Delta+2}\varepsilon^{(h+1)}_{h-\Delta+3}\cdots\varepsilon^{(h+1)}_{h}$, $\varepsilon^{(h+1)}_{h+1-\Delta+i}$ is either 0, 1, or 2, and at most one $\varepsilon^{(h+1)}_{h+1-\Delta+k} = 2$, for some $0 < k < \Delta$, in which case $\varepsilon^{(h+1)}_{h+1-\Delta+i} = 0$, for all $k < i < \Delta$.

First we show that Fringe Diff(h) $\leqslant (\Delta^2 - \Delta + 2)/2$. Since $\varepsilon^{(h)}_{h-\Delta+i} \geqslant 0$, we have

$$\begin{aligned}\text{Fringe Diff}(h) &= \sum_{i=1}^{\Delta-1} (\varepsilon^{(h+1)}_{h+1-\Delta+i} - \varepsilon^{(h)}_{h-\Delta+i})(\Delta - i)\\ &\leqslant \sum_{i=1}^{\Delta-1} \varepsilon^{(h+1)}_{h+1-\Delta+i}(\Delta - i).\end{aligned}$$

It may be that $\varepsilon^{(h+1)}_{h+1-\Delta+k} = 2$, for some k, $1 \leqslant k < \Delta$, or it may be that no $\varepsilon^{(h+1)}_{h+1-\Delta+k} = 2$. We find the maximum value for Fringe Diff(h) for each of these cases separately, then find the overall maximum value for Fringe Diff(h).

If no $\varepsilon^{(h+1)}_{h+1-\Delta+k} = 2$, then Fringe Diff($h$) is as large as possible if all $\varepsilon^{(h+1)}_{h+1-\Delta+i} = 1$. Therefore, if no $\varepsilon^{(h+1)}_{h+1-\Delta+k} = 2$, then

$$\text{Fringe Diff}(h) \leqslant \sum_{i=1}^{\Delta-1} (\Delta - i) = \frac{(\Delta - 1)\Delta}{2}.$$

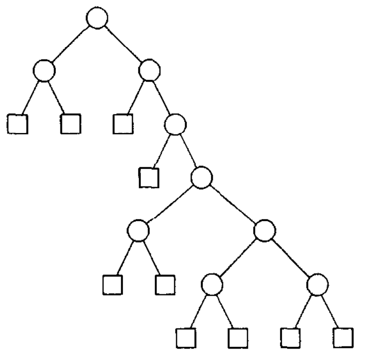

Fig. 4. A maximum path length binary tree with $\Delta=4$, $h=6$ and $N=9$.

If some $\varepsilon^{(h+1)}_{h+1-\Delta+k}=2$, then Fringe Diff($h$) is maximized if all $\varepsilon^{(h+1)}_{h+1-\Delta+j}=1$, for all j, $1\leqslant j<k$; that is, Fringe Diff(h) is maximized if $\varepsilon^{(h+1)}_{h-\Delta+2}\varepsilon^{(h+1)}_{h-\Delta+3}\cdots\varepsilon^{(h+1)}_{h}$ is $1,1,\ldots,1,2,0,0,\ldots,0$. In this case,

$$\begin{aligned}\text{Fringe Diff}(h) &\leqslant \sum_{i=1}^{k-1}(\Delta-i)+2(\Delta-k)\\ &=\frac{(\Delta-1)\Delta}{2}-\frac{(\Delta-k)(\Delta-k-3)}{2}.\end{aligned}$$

The minimum value of $(\Delta-x)(\Delta-x-3)$ is at $x=\Delta-3/2$. Since k is an integer, Fringe Diff(h) is maximized when $k=\Delta-1$ or $k=\Delta-2$. When $k=\Delta-1$ or $k=\Delta-2$,

$$\text{Fringe Diff}(h)=\frac{(\Delta-1)\Delta}{2}-\frac{-2}{2}=\frac{\Delta^2-\Delta+2}{2}.$$

Therefore, Fringe Diff(h)$\leqslant(\Delta^2-\Delta+2)/2$.

Similarly, Fringe Diff(h) is minimized if $\varepsilon^{(h+1)}_{h+1-\Delta+i}=0$ and $\varepsilon^{(h)}_{h-\Delta+1}\varepsilon^{(h)}_{h-\Delta+2}\cdots\varepsilon^{(h)}_{h-1}$ is $1,1,\ldots,1,2,0$ or $1,1,\ldots,1,2$. Therefore, we have

$$-(\Delta^2-\Delta+2)/2\leqslant\text{Fringe Diff}(h). \qquad \square$$

Now let us take a closer look at MaxPL(Δ,h,N) when $\Delta>\log_2(N+1)$. When $\Delta>\log_2(N+1)$, since MaxPL(Δ,h,N) has N internal nodes, there are not enough internal nodes to have even one Bin(Δ) subtree rooted on Level $h-\Delta$, since we must have a Bin($h-\Delta$) prefix above Level $h-\Delta$. For example, Fig. 4 shows a maximum path length binary tree with $\Delta=4$, $h=6$, and $N=9$; since $\Delta>\log_2(N+1)$, there are not enough nodes for a Bin(4) subtree rooted on Level 2. Therefore, the value of r_h is fixed and we can say more about the fringe profile of MaxPL(Δ,h,N), allowing us to compute a narrower range for Fringe Diff(h).

First we find the value of r_h when $\Delta>\log_2(N+1)$.

Lemma 11. *If $\Delta > \log_2(N+1)$, then $r_h = 1$; that is, there is exactly one internal node on Level $h-\Delta$ of a tree in* MaxPL(Δ,h,N), *for any h, $h_{\min} \leqslant h \leqslant h_{\max}$.*

Proof. If $\Delta > \log_2(N+1)$, then $N = 2^{\log_2(N+1)} - 1 < 2^\Delta - 1$. Therefore,

$$r_h = \left\lfloor \frac{N - 2^{h-\Delta}}{2^\Delta - 1} \right\rfloor + 1 = 1. \qquad \square$$

Next we derive more information about the fringe profile of MaxPL(Δ,h,N) when $\Delta > \log_2(N+1)$.

Lemma 12. *If $\Delta > \log_2(N+1)$, then $\varepsilon^{(h)}_{h-\Delta+i} = 1$, for $1 \leqslant i \leqslant \Delta - 1 - \lceil \log_2(N+1) \rceil$.*

Proof. If $\Delta > \log_2(N+1)$, then $\Delta \geqslant \lceil \log_2(N+1) \rceil$, since Δ is an integer. By Lemma 11, if $\Delta > \log_2(N+1)$, then $r_h = 1$, for all h, $h_{\min} \leqslant h \leqslant h_{\max}$. Recall that $\varepsilon^{(h)}_{h-\Delta+1}\varepsilon^{(h)}_{h-\Delta+2}\cdots\varepsilon^{(h)}_{h-1}$ is the canonical pseudo-binary representation of the number $\mathrm{num}_h = 2^{h-\Delta} + r_h(2^\Delta - 1) - N - 1$. Since $r_h = 1$, $\mathrm{num}_h = 2^{h-\Delta} + 2^\Delta - 1 - N - 1$.

The height h is bounded from below by $h_{\min} = \log_2(N + 2^\Delta)$. Since the height must be an integer, the minimum height is $h = \lceil \log_2(N + 2^\Delta) \rceil$. Since $\Delta > \log_2(N+1)$, we have $2^\Delta > N + 1$. Therefore, $2^\Delta < 2^\Delta + N < 2^{\Delta+1}$ or $\Delta < \log_2(N + 2^\Delta) < \Delta + 1$. If $h = \lceil \log_2(N + 2^\Delta) \rceil$, then $h = \Delta + 1$. Therefore, the minimum value of num_h is

$$\begin{aligned}\mathrm{num}_h &\geqslant 2^{\Delta+1-\Delta} + 2^\Delta - 1 - N - 1\\ &= 2^\Delta - N.\end{aligned}$$

The height h is bounded from above by $h_{\max} = \Delta + \log_2(N + 1 - \Delta)$, so the maximum value of num_h is

$$\begin{aligned}\mathrm{num}_h &\leqslant 2^{\log_2(N+1-\Delta)} + 2^\Delta - N - 2\\ &= 2^\Delta - \Delta - 1.\end{aligned}$$

By Lemma 2, $\varepsilon^{(h)}_{h-\Delta+i} = 1$, for $1 \leqslant i \leqslant \Delta - 1 - \lceil \log_2(N+1) \rceil$, if and only if $2^{m+1} - 2^k - m + k - 1 \leqslant \mathrm{num}_h \leqslant 2^{m+1} - m + k - 3$, where $m = \Delta - 1$ and $k = \lceil \log_2(N+1) \rceil + 1$. In other words, $\varepsilon^{(h)}_{h-\Delta+i} = 1$, for $1 \leqslant i \leqslant \Delta - 1 - \lceil \log_2(N+1) \rceil$, if and only if $2^\Delta - 2^{\lceil \log_2(N+1) \rceil + 1} - \Delta + \lceil \log_2(N+1) \rceil + 1 \leqslant \mathrm{num}_h \leqslant 2^\Delta - \Delta + \lceil \log_2(N+1) \rceil - 1$. Since we have $2^\Delta - N \leqslant \mathrm{num}_h \leqslant 2^\Delta - \Delta - 1$, we show that

$$2^\Delta - 2^{\lceil \log_2(N+1) \rceil + 1} - \Delta + \lceil \log_2(N+1) \rceil + 1 \leqslant 2^\Delta - N$$

and that

$$2^\Delta - \Delta - 1 \leqslant 2^\Delta - \Delta + \lceil \log_2(N+1) \rceil - 1.$$

First we show that

$$2^\Delta - 2^{\lceil \log_2(N+1) \rceil + 1} - \Delta + \lceil \log_2(N+1) \rceil + 1 \leqslant 2^\Delta - N.$$

Since $\log_2(N+1) \leqslant \lceil \log_2(N+1) \rceil$, we have $2(N+1) \leqslant 2^{\lceil \log_2(N+1) \rceil + 1}$. Therefore,

$$\begin{aligned}
&2^{\Delta} - 2^{\lceil \log_2(N+1) \rceil + 1} - \Delta + \lceil \log_2(N+1) \rceil + 1 \\
&\leqslant 2^{\Delta} - 2(N+1) - \Delta + \lceil \log_2(N+1) \rceil + 1 \\
&= 2^{\Delta} - 2N - \Delta + \lceil \log_2(N+1) \rceil - 1.
\end{aligned}$$

Since $\log_2(N+1) < \Delta$, we have $\lceil \log_2(N+1) \rceil - 1 < \Delta$. Therefore,

$$2^{\Delta} - 2^{\lceil \log_2(N+1) \rceil + 1} - \Delta + \lceil \log_2(N+1) \rceil + 1 < 2^{\Delta} - 2N.$$

Since $N > 4$, we have $N < 2N$. Therefore,

$$2^{\Delta} - 2^{\lceil \log_2(N+1) \rceil + 1} - \Delta + \lceil \log_2(N+1) \rceil + 1 < 2^{\Delta} - N.$$

Next we show that

$$2^{\Delta} - \Delta - 1 \leqslant 2^{\Delta} - \Delta + \lceil \log_2(N+1) \rceil - 1.$$

Since $4 < N$, we have $\lceil \log_2(N+1) \rceil > 0$. Therefore,

$$2^{\Delta} - \Delta - 1 \leqslant 2^{\Delta} - \Delta + \lceil \log_2(N+1) \rceil - 1.$$

Therefore, if $\Delta > \log_2(N+1)$, then $\varepsilon^{(h)}_{h-\Delta+i} = 1$, for $1 \leqslant i \leqslant \Delta - 1 - \lceil \log_2(N+1) \rceil$. □

Next we use the extra information known about MaxPL(Δ, h, N) when $\Delta > \log_2(N+1)$ to give a tighter bound on Fringe Diff(h) when $\Delta > \log_2(N+1)$.

Lemma 13. *If* $\Delta > \log_2(N+1)$, *then*

$$\begin{aligned}
&-\frac{\lceil \log_2(N+1) \rceil^2 + \lceil \log_2(N+1) \rceil + 2}{2} \\
&\leqslant \text{Fringe Diff}(h) \\
&\leqslant \frac{\lceil \log_2(N+1) \rceil^2 + \lceil \log_2(N+1) \rceil + 2}{2}.
\end{aligned}$$

Proof. By Lemma 12, since $\log_2(N+1) < \Delta$, we have $\varepsilon^{(h)}_{h-\Delta+i} = 1$ and $\varepsilon^{(h+1)}_{h+1-\Delta+i} = 1$, for $1 \leqslant i \leqslant \Delta - 1 - \lceil \log_2(N+1) \rceil$. Therefore,

$$\begin{aligned}
\text{Fringe Diff}(h) &= \sum_{i=1}^{\Delta-1} (\Delta - i)(\varepsilon^{(h+1)}_{h+1-\Delta+i} - \varepsilon^{(h)}_{h-\Delta+i}) \\
&= \sum_{i=\Delta-\lceil \log_2(N+1) \rceil}^{\Delta-1} (\Delta - i)(\varepsilon^{(h+1)}_{h+1-\Delta+i} - \varepsilon^{(h)}_{h-\Delta+i}).
\end{aligned}$$

We proceed with arguments similar to those used in the proof of Lemma 10. The maximum value for Fringe Diff(h) is achieved if $\varepsilon^{(h)}_{h-\Delta+i} = 0$, for $\Delta - \lceil \log_2(N+1) \rceil \leqslant i \leqslant \Delta - 1$.

If $\varepsilon^{(h+1)}_{h+1-\Delta+i}$ is 0 or 1, but not 2, for all $\Delta - \lceil \log_2(N+1) \rceil \leqslant i \leqslant \Delta - 1$, then Fringe Diff($h$) is maximized if $\varepsilon^{(h+1)}_{h+1-\Delta+i} = 1$, for all $\Delta - \lceil \log_2(N+1) \rceil \leqslant i \leqslant \Delta - 1$. Therefore,

$$\begin{aligned}
\text{Fringe Diff}(h) &= \sum_{i=\Delta-\lceil \log_2(N+1)\rceil}^{\Delta-1} (\Delta - i)(\varepsilon^{(h+1)}_{h+1-\Delta+i} - \varepsilon^{(h)}_{h-\Delta+i}) \\
&\leqslant \sum_{i=\Delta-\lceil \log_2(N+1)\rceil}^{\Delta-1} (\Delta - i) \\
&= \frac{\lceil \log_2(N+1)\rceil(\lceil \log_2(N+1)\rceil + 1)}{2}.
\end{aligned}$$

If $\varepsilon^{(h+1)}_{h+1-\Delta+k} = 2$, where $k = \Delta - \lceil \log_2(N+1) \rceil$, then $\varepsilon^{(h+1)}_{h+1-\Delta+i} = 0$, for $k < i \leqslant \Delta - 1$, and

$$\text{Fringe Diff}(h) = 2\lceil \log_2(N+1)\rceil.$$

If $\varepsilon^{(h+1)}_{h+1-\Delta+k} = 2$, for some $\Delta - \lceil \log_2(N+1) \rceil < k \leqslant \Delta - 1$, then $\varepsilon^{(h+1)}_{h+1-\Delta+i} = 0$, for $k < i \leqslant \Delta - 1$. Furthermore, $\varepsilon^{(h+1)}_{h+1-\Delta+i}$ is either 0 or 1, for $\Delta - \lceil \log_2(N+1) \rceil \leqslant i < k$. Fringe Diff($h$) is maximized if $\varepsilon^{(h+1)}_{h+1-\Delta+i} = 1$, for $\Delta - \lceil \log_2(N+1) \rceil \leqslant i < k$. Therefore, if $\varepsilon^{(h+1)}_{h+1-\Delta+k} = 2$, for some $\Delta - \lceil \log_2(N+1) \rceil < k \leqslant \Delta - 1$, then

$$\begin{aligned}
\text{Fringe Diff}(h) &= \sum_{i=\Delta-\lceil \log_2(N+1)\rceil}^{\Delta-1} (\Delta - i)(\varepsilon^{(h+1)}_{h+1-\Delta+i} - \varepsilon^{(h)}_{h-\Delta+i}) \\
&\leqslant \sum_{i=\Delta-\lceil \log_2(N+1)\rceil}^{k-1} (\Delta - i)(1-0) + (\Delta - k)(2-0) \\
&= \frac{\lceil \log_2(N+1)\rceil(\lceil \log_2(N+1)\rceil + 1)}{2} - \frac{(\Delta-k)(\Delta-k-3)}{2}.
\end{aligned}$$

Since the minimum of $(\Delta - x)$ $(\Delta - x - 3)$ is at $x = \Delta - 3/2$, Fringe Diff(h) is maximized if $k = \Delta - 1$ or if $k = \Delta - 2$. Therefore, if $\varepsilon^{(h+1)}_{h+1-\Delta+k} = 2$, for some $\Delta - \lceil \log_2(N + 1) \rceil < k \leqslant \Delta - 1$, then

$$\begin{aligned}
\text{Fringe Diff}(h) &\leqslant \frac{\lceil \log_2(N+1)\rceil(\lceil \log_2(N+1)\rceil + 1)}{2} - \frac{(\Delta-k)(\Delta-k-3)}{2} \\
&\leqslant \frac{\lceil \log_2(N+1)\rceil(\lceil \log_2(N+1)\rceil + 1)}{2} - \frac{-2}{2} \\
&= \frac{\lceil \log_2(N+1)\rceil^2 + \lceil \log_2(N+1)\rceil + 2}{2}.
\end{aligned}$$

Therefore, if $\log_2(N+1) < \Delta$, then

$$\text{Fringe Diff}(h) \leqslant \frac{\lceil \log_2(N+1)\rceil^2 + \lceil \log_2(N+1)\rceil + 2}{2}.$$

Table 2
$N \leqslant 43$ and Δ such that MaxPL(Δ, N) contains trees of height $h_{\Delta,N}-1$ or $h_{\Delta,N}+1$

N	Δ	Height(s) of trees in MaxPL(Δ,N)	N	Δ	Height(s) of trees in MaxPL(Δ,N)
6	3	$h_{\Delta,N}-1=4$ and $h_{\Delta,N}=5$	29	7	$h_{\Delta,N}-1=9$
8	4	$h_{\Delta,N}-1=5$ and $h_{\Delta,N}=6$	29	15	$h_{\Delta,N}+1=17$
9	4	$h_{\Delta,N}-1=5$ and $h_{\Delta,N}=6$	30	16	$h_{\Delta,N}+1=18$
10	5	$h_{\Delta,N}-1=6$	31	16	$h_{\Delta,N}+1=18$
12	3	$h_{\Delta,N}=5$ and $h_{\Delta,N}+1=6$	31	17	$h_{\Delta,N}+1=19$
13	7	$h_{\Delta,N}=8$ and $h_{\Delta,N}+1=9$	32	8	$h_{\Delta,N}-1=10$ and $h_{\Delta,N}=11$
15	8	$h_{\Delta,N}+1=10$	32	16	$h_{\Delta,N}-1=17$ and $h_{\Delta,N}=18$
16	8	$h_{\Delta,N}-1=9$ and $h_{\Delta,N}=10$	32	17	$h_{\Delta,N}=18$ and $h_{\Delta,N}+1=19$
16	9	$h_{\Delta,N}=10$ and $h_{\Delta,N}+1=11$	32	18	$h_{\Delta,N}=19$ and $h_{\Delta,N}+1=20$
17	4	$h_{\Delta,N}-1=6$	33	16	$h_{\Delta,N}-1=17$ and $h_{\Delta,N}=18$
17	8	$h_{\Delta,N}-1=9$ and $h_{\Delta,N}=10$	34	4	$h_{\Delta,N}-1=7$
18	4	$h_{\Delta,N}-1=6$ and $h_{\Delta,N}=7$	34	17	$h_{\Delta,N}-1=18$
18	9	$h_{\Delta,N}-1=10$	36	9	$h_{\Delta,N}-1=11$
20	2	$h_{\Delta,N}=5$ and $h_{\Delta,N}+1=6$	36	18	$h_{\Delta,N}-1=19$ and $h_{\Delta,N}=20$
20	10	$h_{\Delta,N}-1=11$ and $h_{\Delta,N}=12$	37	9	$h_{\Delta,N}-1=11$
22	6	$h_{\Delta,N}=8$ and $h_{\Delta,N}+1=9$	38	9	$h_{\Delta,N}-1=11$ and $h_{\Delta,N}=12$
22	11	$h_{\Delta,N}-1=12$ and $h_{\Delta,N}=13$	38	19	$h_{\Delta,N}-1=20$ and $h_{\Delta,N}=21$
23	6	$h_{\Delta,N}+1=9$	39	5	$h_{\Delta,N}=8$ and $h_{\Delta,N}+1=9$
23	11	$h_{\Delta,N}-1=12$ and $h_{\Delta,N}=13$	39	19	$h_{\Delta,N}-1=20$ and $h_{\Delta,N}=21$
24	12	$h_{\Delta,N}-1=13$	40	5	$h_{\Delta,N}+1=9$
25	7	$h_{\Delta,N}=9$ and $h_{\Delta,N}+1=10$	40	20	$h_{\Delta,N}-1=21$
26	3	$h_{\Delta,N}=6$ and $h_{\Delta,N}+1=7$	42	2	$h_{\Delta,N}-1=6$ and $h_{\Delta,N}=7$
27	3	$h_{\Delta,N}-1=6$ and $h_{\Delta,N}=7$	42	11	$h_{\Delta,N}+1=14$
27	14	$h_{\Delta,N}+1=16$	43	11	$h_{\Delta,N}+1=14$
28	7	$h_{\Delta,N}-1=9$	43	22	$h_{\Delta,N}=23$ and $h_{\Delta,N}+1=24$
28	15	$h_{\Delta,N}=16$ and $h_{\Delta,N}+1=17$			

Similarly, Fringe Diff(h) is minimized if $\varepsilon^{(h+1)}_{h+1-\Delta+i}=0$, for all $\Delta-\lceil\log_2(N+1)\rceil\leqslant i\leqslant \Delta-1$, and $\varepsilon^{(h)}_{h-\Delta+k}=2$, for $k=\Delta-1$ or $k=\Delta-2$. Therefore, if $\log_2(N+1)<\Delta$, then

$$-\frac{\lceil\log_2(N+1)\rceil^2+\lceil\log_2(N+1)\rceil+2}{2}\leqslant \text{Fringe Diff}(h). \qquad \Box$$

For small values of N, $N\leqslant 43$, we prove that MaxPL(Δ,N) has height $h_{\Delta,N}-1$, $h_{\Delta,N}$, or $h_{\Delta,N}+1$ by direct computation.

Lemma 14. *When* $4<N\leqslant 43$, MaxPL(Δ,N) *has height* $h_{\Delta,N}-1$, $h_{\Delta,N}$, *or* $h_{\Delta,N}+1$.

Proof. By calculating the path length of MaxPL(Δ,h,N) for each height h such that $h_{\min}\leqslant h\leqslant h_{\max}$, we find that the height of trees in MaxPL(Δ,N) is $h_{\Delta,N}$ except in the cases displayed in Table 2. $\Box$

In the following sections, we assume that $N>43$.

4.2. Main results

In this section, we show that the heights of trees in MaxPL(Δ,N) are $h_{\Delta,N}-1$, $h_{\Delta,N}$, or $h_{\Delta,N}+1$. We first examine PL Diff(h) for $h_{\min}\leqslant h<h_{\Delta,N}-1$, and then we examine PL Diff(h) for $h_{\Delta,N}<h<h_{\max}$.

4.2.1. Heights less than $h_{\Delta,N}-1$

For the first half of the proof that MaxPL(Δ,N) has height $h_{\Delta,N}-1$, $h_{\Delta,N}$, or $h_{\Delta,N}+1$, we show that MaxPL(Δ,h,N) increases as h increases from $\lceil h_{\min}\rceil$ to $h_{\Delta,N}-1$ by showing that PL Diff$(h)>0$, for all $h_{\min}\leqslant h<h_{\Delta,N}-1$. First we derive a lower bound for PL Diff(h).

Theorem 15. *For* $h_{\min}\leqslant h<h_{\Delta,N}-1$,

$$\text{PL Diff}(h)>\frac{N+1}{2}-\Delta-\text{Fringe Diff}(h).$$

Proof. Since

$$\text{PL Diff}(h)=N+1-\Delta 2^{h-\Delta}+\Delta(r_{h+1}-r_h)-\text{Fringe Diff}(h),$$

we show that $N+1-\Delta 2^{h-\Delta}+\Delta(r_{h+1}-r_h)>(N+1)/2-\Delta$, for $h_{\min}\leqslant h<h_{\Delta,N}-1$. By Lemma 9,

$$r_{h+1}-r_h>-\frac{2^{h-\Delta}}{2^{\Delta}-1}-1,$$

we have

$$\begin{aligned}
&N+1-\Delta 2^{h-\Delta}+\Delta(r_{h+1}-r_h)\\
&\quad>N+1-\Delta 2^{h-\Delta}+\Delta\left(-\frac{2^{h-\Delta}}{2^{\Delta}-1}-1\right)\\
&\quad=N+1-\Delta\frac{2^{h}}{2^{\Delta}-1}-\Delta.
\end{aligned}$$

Since

$$h\leqslant h_{\Delta,N}-2=\left\lceil\log_2\frac{(N+1)(2^{\Delta}-1)}{\Delta}\right\rceil-2<\log_2\frac{(N+1)(2^{\Delta}-1)}{\Delta}-1,$$

we have

$$\begin{aligned}
&N+1-\Delta 2^{h-\Delta}+\Delta(r_{h+1}-r_h)\\
&\quad>N+1-\Delta\frac{2^{h}}{2^{\Delta}-1}-\Delta\\
&\quad>N+1-\Delta\frac{2^{\log_2((N+1)(2^{\Delta}-1)/\Delta)-1}}{2^{\Delta}-1}-\Delta
\end{aligned}$$

$$= N + 1 - \Delta \frac{(N+1)(2^{\Delta} - 1)}{\Delta(2^{\Delta} - 1)2} - \Delta$$

$$= N + 1 - \frac{N+1}{2} - \Delta$$

$$= \frac{N+1}{2} - \Delta. \qquad \square$$

We consider the case when $\Delta \leqslant \log_2(N+1)$ and the case when $\Delta > \log_2(N+1)$ separately. First we consider $\Delta \leqslant \log_2(N+1)$.

Theorem 16. *If* $\Delta \leqslant \log_2(N+1)$ *and* $h_{\min} \leqslant h < h_{\Delta,N} - 1$, *then* PL Diff$(h) > 0$.

Proof. By Theorem 15, since $h_{\min} \leqslant h < h_{\Delta,N} - 1$, we have

$$\text{PL Diff}(h) > \frac{N+1}{2} - \Delta - \text{Fringe Diff}(h).$$

By Lemma 10,

$$\text{Fringe Diff}(h) \leqslant (\Delta^2 - \Delta + 2)/2.$$

Combining the two relations, we have

$$\text{PL Diff}(h) > \frac{N+1}{2} - \frac{\Delta^2 + \Delta + 2}{2}.$$

Since $\Delta \leqslant \log_2(N+1)$, we have

$$\text{PL Diff}(h) > \frac{N+1}{2} - \frac{(\log_2(N+1))^2 + \log_2(N+1) + 2}{2}.$$

When $N > 31$, we have $N+1 > (\log_2(N+1))^2 + \log_2(N+1) + 2$. Since we only consider $N > 43$, PL Diff$(h) > 0$. $\square$

Now we consider the case when $\Delta > \log_2(N+1)$.

Theorem 17. *If* $\Delta > \log_2(N+1)$ *and* $h_{\min} \leqslant h < h_{\Delta,N} - 1$, *then* PL Diff$(h) > 0$.

Proof. By Lemma 11, if $\Delta > \log_2(N+1)$, then $r_h = r_{h+1} = 1$, so $r_{h+1} - r_h = 0$. Therefore, PL Diff$(h) = N + 1 - \Delta 2^{h-\Delta}$ – Fringe Diff(h). Since $h \leqslant \lceil \log_2(N+1)(2^{\Delta} - 1)/\Delta \rceil - 2 < \log_2(N+1)(2^{\Delta} - 1)/\Delta - 1$, we have

$$\begin{aligned}
\text{PL Diff}(h) &> N + 1 - \Delta 2^{\log_2(N+1)(2^{\Delta}-1)/\Delta - 1 - \Delta} - \text{Fringe Diff}(h) \\
&= N + 1 - \Delta \frac{(N+1)(2^{\Delta} - 1)}{\Delta} \frac{1}{2^{\Delta} \times 2} - \text{Fringe Diff}(h) \\
&= (N+1)\left[1 - \frac{2^{\Delta} - 1}{2^{\Delta}} \frac{1}{2}\right] - \text{Fringe Diff}(h).
\end{aligned}$$

Since $(2^{\Delta}-1)/2^{\Delta}<1$, we have

$$1-\frac{2^{\Delta}-1}{2^{\Delta}}\frac{1}{2}>1-\frac{1}{2}=\frac{1}{2}.$$

Thus,

$$\text{PL Diff}(h)>\frac{N+1}{2}-\text{Fringe Diff}(h).$$

By Lemma 13, since $\Delta>\log_2(N+1)$, we have

$$\text{Fringe Diff}(h)\leqslant\frac{\lceil\log_2(N+1)\rceil^2+\lceil\log_2(N+1)\rceil+2}{2}.$$

Therefore,

$$\text{PL Diff}(h)>\frac{N+1}{2}-\frac{\lceil\log_2(N+1)\rceil^2+\lceil\log_2(N+1)\rceil+2}{2}.$$

For $N\geqslant 43$, we have $N+1\geqslant\lceil\log_2(N+1)\rceil^2+\lceil\log_2(N+1)\rceil+2$. Therefore, if $\Delta>\log_2(N+1)$ and $N\geqslant 43$, we have PL Diff$(h)>0$. □

4.2.2. Heights greater than $h_{\Delta,N}$

For the second half of the proof that MaxPL(Δ,N) has height $h_{\Delta,N}-1$, $h_{\Delta,N}$, or $h_{\Delta,N}+1$, we show that MaxPL(Δ,h,N) decreases as h increases from $h_{\Delta,N}+1$ to $\lfloor h_{\max}\rfloor$ by showing that PL Diff$(h)<0$, for all $h_{\Delta,N}<h<h_{\max}$. First we derive a lower bound for PL Diff(h).

Theorem 18. *For $h_{\Delta,N}<h<h_{\max}$,*

$$\text{PL Diff}(h)<-(N+1)+\Delta-\text{Fringe Diff}(h).$$

Proof. Since

$$\text{PL Diff}(h)=N+1-\Delta 2^{h-\Delta}+\Delta(r_{h+1}-r_h)-\text{Fringe Diff}(h),$$

we show that $N+1-\Delta 2^{h-\Delta}+\Delta(r_{h+1}-r_h)<-(N+1)+\Delta$, for $h_{\Delta,N}<h<h_{\max}$. By Lemma 9,

$$r_{h+1}-r_h<-\frac{2^{h-\Delta}}{2^{\Delta}-1}+1,$$

we have

$$\begin{aligned}&N+1-\Delta 2^{h-\Delta}+\Delta(r_{h+1}-r_h)\\&<N+1-\Delta 2^{h-\Delta}+\Delta\left(-\frac{2^{h-\Delta}}{2^{\Delta}-1}+1\right)\\&=N+1-\Delta\frac{2^{h}}{2^{\Delta}-1}+\Delta.\end{aligned}$$

Since

$$h \geqslant \left\lceil \log_2 \frac{(N+1)(2^{\Delta}-1)}{\Delta} \right\rceil + 1 \geqslant \log_2 \frac{(N+1)(2^{\Delta}-1)}{\Delta} + 1,$$

we have

$$\begin{aligned}
&N+1-\Delta 2^{h-\Delta}+\Delta(r_{h+1}-r_h)\\
&< N+1-\Delta\frac{2^h}{2^{\Delta}-1}+\Delta\\
&\leqslant N+1-\Delta\frac{2^{\log_2((N+1)(2^{\Delta}-1)/\Delta)+1}}{2^{\Delta}-1}+\Delta\\
&= N+1-\Delta\frac{2(N+1)(2^{\Delta}-1)}{\Delta(2^{\Delta}-1)}+\Delta\\
&= -(N+1)+\Delta. \qquad \square
\end{aligned}$$

We consider the case when $\Delta \leqslant \log_2(N+1)$ and the case when $\Delta > \log_2(N+1)$ separately. First we consider $\Delta \leqslant \log_2(N+1)$.

Theorem 19. *When* $\Delta \leqslant \log_2(N+1)$ *and* $h_{\Delta,N} < h < h_{\max}$, *we have* PL Diff$(h) < 0$.

Proof. By Theorem 18, PL Diff$(h) < -(N+1)+\Delta-$Fringe Diff(h). By Lemma 10, we have

$$-\frac{\Delta^2-\Delta+2}{2} \leqslant \text{Fringe Diff}(h).$$

Therefore,

$$\text{PL Diff}(h) < -(N+1)+\Delta+\frac{\Delta^2-\Delta+2}{2} = -(N+1)+\frac{\Delta^2+\Delta+2}{2}.$$

Since $\Delta \leqslant \log_2(N+1)$, we have

$$\text{PL Diff}(h) < -(N+1)+\frac{(\log_2(N+1))^2+\log_2(N+1)+2}{2}.$$

Since $(N+1) > ((\log_2(N+1))^2+\log_2(N+1)+2)/2$, for $N \geqslant 5$, we have PL Diff(h) < 0. $\square$

Finally, we consider $\Delta > \log_2(N+1)$.

Theorem 20. *When* $\Delta > \log_2(N+1)$ *and* $h_{\Delta,N} < h < h_{\max}$, *we have* PL Diff$(h) < 0$.

Proof. Since $\Delta > \log_2(N+1)$, by Lemma 11, $r_h = r_{h+1} = 1$, so $r_{h+1} - r_h = 0$. Therefore PL Diff$(h) = N+1-\Delta 2^{h-\Delta}-$Fringe Diff$(h)$. Since $h_{\Delta,N} < h$,

$$\Delta 2^{\lceil \log_2(N+1)(2^{\Delta}-1)/\Delta \rceil + 1 - \Delta} \leqslant \Delta 2^{h-\Delta}.$$

Since $\Delta > 1$, $2^\Delta - 1$ is an odd number and $\Delta < 2^\Delta - 1$. Therefore $(N+1)(2^\Delta-1)/\Delta$ is not a power of two,

$$\log_2 \frac{(N+1)(2^\Delta-1)}{\Delta} < \left\lceil \log_2 \frac{(N+1)(2^\Delta-1)}{\Delta} \right\rceil,$$

and we have

$$\Delta 2^{\log_2 \frac{(N+1)(2^\Delta-1)}{\Delta} + 1 - \Delta} = 2\frac{(N+1)(2^\Delta-1)}{2^\Delta} < \Delta 2^{h-\Delta}.$$

Substituting the lower bound for $\Delta 2^{h-\Delta}$ into the formula for PL Diff(h), we get

$$\text{PL Diff}(h) < N+1-2\frac{(N+1)(2^\Delta-1)}{2^\Delta} - \text{Fringe Diff}(h).$$

Since $2 \leqslant \Delta$, we have $3/4 \leqslant (2^\Delta - 1)/2^\Delta$. Therefore,

$$\text{PL Diff}(h) < -\frac{N+1}{2} - \text{Fringe Diff}(h).$$

By Lemma 13, since $\Delta > \log_2(N+1)$, we have

$$-\frac{\lceil \log_2(N+1) \rceil^2 + \lceil \log_2(N+1) \rceil + 2}{2} \leqslant \text{Fringe Diff}(h).$$

Therefore,

$$\text{PL Diff}(h) < -\frac{N+1}{2} + \frac{\lceil \log_2(N+1) \rceil^2 + \lceil \log_2(N+1) \rceil + 2}{2}.$$

For $N \geqslant 43$, we have $N+1 \geqslant \lceil \log_2(N+1) \rceil^2 + \lceil \log_2(N+1) \rceil + 2$. Therefore, if $\Delta > \log_2(N+1)$ and $N \geqslant 43$, we have PL Diff(h) < 0. □

4.3. Summary

We now pull all the threads together to obtain the desired result:

Theorem 21. *For $\Delta > 1$ and $N > 4$, trees in* MaxPL(Δ, N) *have heights $h_{\Delta,N} - 1$ or $h_{\Delta,N}$ or $h_{\Delta,N} + 1$.*

Proof. By Lemma 14, if $N \leqslant 43$, then the trees in MaxPL(Δ, N) have heights $h_{\Delta,N} - 1$, $h_{\Delta,N}$, or $h_{\Delta,N} + 1$. Thus, without loss of generality, we can assume that $N \geqslant 43$.

Next we examine $\Delta \leqslant \log_2(N+1)$. By Theorem 16 and Theorem 19, if $\Delta \leqslant \log_2(N+1)$ and $N \geqslant 43$, then the path lengths of trees in MaxPL(Δ, h, N) increase as the height h increases from $h_{\min}$ to $h_{\Delta,N} - 1$ and decrease as the height h increases from $h_{\Delta,N} + 1$ to $h_{\max}$. Therefore, the heights of trees in MaxPL(Δ, N) are $h_{\Delta,N} - 1$, $h_{\Delta,N}$, or $h_{\Delta,N} + 1$.

Finally, we examine $\Delta > \log_2(N+1)$. By Theorem 17 and Theorem 20, if $\Delta > \log_2(N+1)$ and $N > 43$, then the path lengths of trees in MaxPL(Δ, h, N) increase as the height h increases from $h_{\min}$ to $h_{\Delta,N} - 1$ and decrease as the height increases from

$h_{\Delta,N}+1$ to $h_{\max}$. Therefore, the trees in MaxPL(Δ,N) have heights $h_{\Delta,N}-1$, $h_{\Delta,N}$, or $h_{\Delta,N}+1$. $\square$

Thus, the maximum path length of binary trees with fringe thickness Δ and size N is

$$(N+1)h-\Delta(2^{h-\Delta}-r_h)-\sum_{i=1}^{\Delta-1}(\Delta-i)\,\varepsilon_{h-\Delta+i},$$

where $r_h=\lfloor(N-2^{h-\Delta})/(2^{\Delta}-1)\rfloor+1$, $\varepsilon_{h-\Delta+1}\varepsilon_{h-\Delta+2}\cdots\varepsilon_{h-1}$ is the canonical pseudo-binary representation of $\mathrm{num}_h=2^{h-\Delta}+r_h(2^{\Delta}-1)-N-1$, and $h=h_{\Delta,N}-1$ or $h=h_{\Delta,N}$ or $h=h_{\Delta,N}+1$.

5. Conclusion

As Lemma 14 shows, the trees in MaxPL(Δ,N) do not necessarily have height $h_{\Delta,N}$. We must examine at most three heights, $h_{\Delta,N}-1$, $h_{\Delta,N}$, and $h_{\Delta,N}+1$, in order to find the trees in MaxPL(Δ,N). This result leaves open questions on the exact heights of trees in MaxPL(Δ,N). For what fringe thicknesses and sizes is $h_{\Delta,N}$ the height of all trees in MaxPL(Δ,N)? When is $h_{\Delta,N}$ not the height of any tree in MaxPL(Δ,N)? Do the trees in MaxPL(Δ,N) have at most two different heights? In other words, is there some fringe thickness Δ and size N for which MaxPL(Δ,N) contain trees of height $h_{\Delta,N}-1$, $h_{\Delta,N}$, and $h_{\Delta,N}+1$?

Another interesting open problem is to find the maximum and minimum path lengths of multi-way trees of a given fringe thickness and size.

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